

Acoustic phase lenses in superfluid helium as models of composite spacetimes in general relativity: Classical and quantum features

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Abstract

In the spirit of the well-known analogy between inviscid fluids and pseudo-Riemannian manifolds we study spherical singular hypersurfaces in the static superfluid. Such hypersurfaces turn out to be the interfaces dividing the superfluid into the pairs of spherical domains, examples of which are “superfluid A - superfluid B” or “impurity - superfluid” phases. It is shown that these shells form the acoustic lenses which are the sonic counterparts of the usual optical ones. The exact equations of motion of the lens interfaces are obtained. Also some quantum aspects of the theory are considered. We calculate energy spectra for bound states of acoustic lenses in dynamical equilibrium, taking into account the analogy to a material shell model of a black hole (we consider the cases of spatial topology of a black hole and a wormhole type).

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It was shown in numerous works that the superfluid phases of ^3He (and perhaps ^4He) can simulate phenomena encountered in gravitation and the standard model of elementary particles. Physics of superfluid ^3He illustrates concepts in quantum field theory and gravity such as: black holes, surface gravity, Hawking radiation, horizons, ergoregions, trapped surfaces [1, 2, 3] (see Ref. [4] for an introduction into recent developments), baryogenesis, vortexes, strings, textures, standard electroweak model (see [5, 6, 11], and references therein), and so on. This turns out to be possible due to the certain analogy between inviscid fluids and pseudo-Riemannian manifolds. The simplest way to show this correspondence is as follows.

The fundamental equations of dynamics of an inviscid fluid are the Euler equation

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla p - \rho \nabla \Phi, \quad (1)$$

and equation of continuity

$$\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (2)$$

where Φ is the potential of an external force (including gravity), \vec{v} is the flow velocity, ρ and p are, respectively, the fluid density and pressure. If one assumes the flow to be locally irrotational then we can introduce the velocity potential ψ , $\vec{v} = -\nabla \psi$. Hence, assuming the barotropic equation of state $\rho(p)$, the Euler equation can be rewritten in the form of the Bernoulli equation [4]

$$-\frac{\partial \psi}{\partial t} + \frac{1}{2}(\nabla \psi)^2 + \int_0^p \frac{dp'}{\rho(p')} + \Phi = 0. \quad (3)$$

We can linearize these equations around some background $\{\rho_0, p_0, \psi_0\}$ to consider the propagation of small fluctuations (sound waves). We suppose $\rho = \rho_0 + \epsilon \rho_1 + o(\epsilon)$, $p = p_0 + \epsilon p_1 + o(\epsilon)$, $\psi = \psi_0 + \epsilon \psi_1 + o(\epsilon)$, the external potential is fixed. Then, linearizing the Euler equation and taking into account the linearized continuity equation, we finally obtain the wave equation describing the propagation of the fluctuation ψ_1

$$\begin{aligned} & -\frac{\partial}{\partial t} \left[\rho_0 \frac{\partial \rho}{\partial p} \left(\frac{\partial \psi_1}{\partial t} + \vec{v}_0 \cdot \nabla \psi_1 \right) \right] \\ & + \nabla \cdot \left[\rho_0 \nabla \psi_1 - \rho_0 \vec{v}_0 \frac{\partial \rho}{\partial p} \left(\frac{\partial \psi_1}{\partial t} + \vec{v}_0 \cdot \nabla \psi_1 \right) \right] = 0. \end{aligned} \quad (4)$$

This equation can be rewritten as the d'Alembert equation in the curved background spacetime

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial \psi_1}{\partial x^\nu} \right) = 0, \quad (5)$$

where $\mu = \{0, i\}$, $x^\mu = \{t, \vec{x}\}$, $g = \det(g_{\mu\nu})$, and the acoustic background metric is

$$ds^2 = \frac{\rho_0}{c} \left[-c^2 dt^2 + \delta_{ij} (dx^i - v_0^i dt)(dx^j - v_0^j dt) \right], \quad (6)$$

where $c = \sqrt{\partial p / \partial \rho}$ is the local speed of sound. Thus, the vorticity-free flow of a zero viscosity fluid can be seen to define a Lorentzian signature metric a curved space-time. Of course, the physical space-time is just the usual Minkowski flat space-time.

The aim of this paper is to study infinitely thin shells in such acoustic space-times as models of physical objects whose thickness is negligible in comparison with a circumference radius (e.g., surfaces of phase domains). A thin shell is thought to be a discontinuity

of the second kind (the density has the delta-like singularity on the shell). Its dynamics is determined by the Lichnerowicz-Darmois-Israel junction conditions: the first quadratic form (metric) is continuous, the second quadratic form (extrinsic curvature) has a finite jump across the shell. Geometrically a shell is described by a three-dimensional closed singular hypersurface, embedded in the four-dimensional space-time and dividing it into two domains: the external (Σ^+) and internal (Σ^-) regions of space-times. Since the classic works [7, 8] the theory of singular hypersurfaces has been widely considered in the literature (see Ref. [9] for details). We describe only some basic properties of timelike hypersurfaces corresponding to dynamical evolution of thin shells now. One considers a singular matter layer Σ described by the three-dimensional space-time with the surface stress-energy tensor of a perfect fluid in the general case

$$S_{ab} = \sigma u_a u_b + \frac{p}{c_\Sigma^2} (u_a u_b + {}^{(3)}g_{ab}), \quad (7)$$

where σ and p are the surface mass-energy density and pressure respectively, u^a is the timelike unit tangent vector, ${}^{(3)}g_{ab}$ is the 3-metric of a shell's surface (in the acoustic sense (6)), c_Σ is the speed of sound in the shell. We suppose metrics of the fluid space-times outside Σ^+ and inside Σ^- of a spherical shell to be flat:

$$ds_\pm^2 = -c_\pm^2 dt^2 + dr^2 + r^2 d\Omega^2, \quad (8)$$

where $d\Omega^2$ is the metric of the unit 2-sphere, c_\pm are the constants of the speed of sound in the space-times Σ^\pm . These metrics correspond to the spherical shell dividing different phase domains inside the motionless homogeneous superfluid (6). It is possible to show that if one uses the shell's proper time τ then the 3-metric of the shell's space-time history is

$${}^{(3)}ds^2 = -c_\Sigma^2 d\tau^2 + R^2 d\Omega^2, \quad (9)$$

where $R(\tau)$ is the shell's radius. As it can be seen from Eqs. (8), (9), we obtain the composite space-time consisting of three regions, Σ_+ , Σ_- , and Σ , characterizing by proper fundamental constants. The space-time domains inside and outside a shell (8) are flat¹ and are characterized only by the fundamental constants of the speed of sound, whereas the three-dimensional shell's space-time can be curved (we mean the curvature in the acoustic sense) and, in addition to the constant c_Σ , the "gravitational" constant γ_Σ may appear as well. The energy conservation law for a shell (which is the shell's interpretation of the integrability condition $S_{b;a}^a = 0$) can be written as

$$c_\Sigma^2 d(\sigma {}^{(3)}g) + p d({}^{(3)}g) = 0, \quad (10)$$

where ${}^{(3)}g = \sqrt{-\det({}^{(3)}g_{ab})} = c_\Sigma R^2 \sin \theta$. In this equation, the first term corresponds to a change in the shell's internal energy, the second term corresponds to the work done by the shell's internal forces.

It is important to note that the analogy between inviscid fluids and pseudo-Riemannian manifolds appears to be justified as yet on the kinematical level only. Thus, the Einstein equations as such have no evident physical sense within the frameworks of inviscid fluid dynamics. Only the (acoustic) metric, the manifold topology, and equations of motion

¹The presence of two flat spacetimes which nevertheless are regarded as different does not contradict to the relativity principle (which means that metrics are equivalent if one can be transformed to another by virtue of coordinate transformations) because here we have three independent domains such that the relativity principle is satisfied inside of every of them *separately*. Besides, it should be pointed out that such a three-regional union spacetime does not appear to be a manifold in the wide-used sense of this word.

(which are the consequence of the Bianchi identity) have direct physical interpretation. However, the above-mentioned junction conditions, strictly speaking, are connected rigidly neither with general relativity nor with the Einstein equations, despite the fact that historically they were first derived in the context of general relativity. They simply represent the procedure of geometrical matching of two Riemannian manifolds across a surface of discontinuity of the second kind, and thus can be supposed independently as equations describing behaviour of an interface between two liquid phase regions. In this connection the words of famous mathematician Kolmogorov that “whole mathematics (therefore, physics too) can be reformulated as geometry” are quite relevant. By imposing the junction conditions

$$(K_b^a)^+ - (K_b^a)^- = 4\pi\sigma(2u^a u_b + {}^{(3)}\delta_b^a),$$

where $(K_{ab})^\pm$ are the extrinsic curvatures of the spherically symmetric singular hypersurface [10] with respect to the external and internal acoustic manifolds Σ^\pm , we obtain the equation of shell’s motion in the form

$$\epsilon_+ \sqrt{1 + (\dot{R}/c_+)^2} - \epsilon_- \sqrt{1 + (\dot{R}/c_-)^2} = -4\pi\zeta\sigma R, \quad (11)$$

where $\dot{R} = dR/d\tau$ is the velocity of a shell, $\epsilon_\pm = \text{sign} \left[\sqrt{1 + (\dot{R}/c_\pm)^2} \right]$ (see below), ζ is a fundamental constant for the shell’s space-time Σ [11], $\zeta = \gamma_\Sigma/c_\Sigma^2$ with the dimensionality $[\zeta] = \text{cm g}^{-1}$.

From Eq. (11) one can see that we obtain a simple but nontrivial object. From the viewpoint of general relativity it has neither mass nor charge nor some other habitual global property. The only its global attribute is to be an interface of two space-times with different fundamental constants. Nevertheless, such shells have the nontrivial local dynamic properties, viz., the proper velocity, tension and mass-energy density (therefore, an equation of state). Moreover, it can easily be seen that the shell’s matter can be assumed to be enough arbitrary one, both highly exotic and ordinary. Once the function $\sigma(R)$ is known then by means of the conservation law (10) we can obtain the equation of state $p = p(\sigma)$.

It should also be noted that a sound, passing from the one space-time (8) to other across the shell-interface, will be refracted, as it happens for light rays in a usual (spherical) lens. Other analogous phenomena, e.g., the spectral factorization or focusing of sound, can appear as well.

The equation of motion (11) together with the equation of state (or, equivalently, with the known function $\sigma(R)$) and choice of the signs ϵ_\pm , completely determines the motion of superfluid shells (interfaces of the acoustic lenses). Therefore, first of all we must specify ϵ_\pm and $\sigma(R)$.

Let us say few words about the topological features of the theory. In general relativity it is well-known [12, 13] that $\epsilon = +1$ if R increases in the outward normal direction to the shell, and $\epsilon = -1$ if R decreases. Thus, under the condition $\epsilon_+ = \epsilon_-$ we have the ordinary (black hole type) shell, and under $\epsilon_+ = -\epsilon_-$ we have the traversable wormhole type shell [14]. The appropriate cases are represented in the table 1 (we assume the surface density σ to be positive), where the shells (i.e., surfaces of the second kind) of ordinary lenses are sonic analogs of the black hole type shells, and shells corresponding to anomalous lenses are counterparts of the wormhole type shells [15]. The superscript “†” denotes the case of ordinary lenses when the notions “outside the shell” and “inside the shell” are reversed (for anomalous, wormhole, lenses such notions are absent *ab initio*).

Below we assume the rate of change of the lens size to be small, $\dot{R} \ll c_\pm$. Otherwise, the disturbances which may occur could be incompatible with the assumed flatness of the

superfluid space-times (8). Following (11), we obtain the equation of motion of the lenses in the form

$$\frac{\mu \dot{R}^2}{2} = \Xi(R), \quad (12)$$

where

$$\begin{aligned} \Xi(R) &= 4\pi\zeta\sigma R - 2\delta, \\ \mu &= c_-^{-2} + (2\delta - 1)c_+^{-2} > 0, \\ \delta &= \begin{cases} 0 & (\text{OL}) \\ 1 & (\text{AL}) \end{cases}, \end{aligned} \quad (13)$$

and we call δ the *parameter of lens anomaly*.

Further, we do not know what is the concrete matter in the shell. However, we can specify the class of the lenses in the dynamical equilibrium (the other lenses will be either growing or decreasing in size and this eventually leads to the vanishing of either of the two phases Σ^\pm). The Taylor expansion of the function $\Xi(R)$ in a small neighborhood of the equilibrium point R_0 yields

$$\Xi(R) = -2\delta + \varepsilon - \frac{k^2}{2}(R - R_0)^2 + o((R - R_0)^2), \quad (14)$$

where ε and k are the constants,

$$\varepsilon = 4\pi\zeta\sigma|_{R=R_0}R_0, \quad k^2 = -4\pi\zeta(\sigma R)''|_{R=R_0}.$$

Then Eq. (12) can be written as the energy conservation law for the harmonic oscillator. Performing the shift $x = R - R_0$, we obtain

$$E = \frac{P^2}{2m} + \frac{2\delta}{\zeta\mu k} + \frac{m\omega^2 x^2}{2}, \quad (15)$$

where

$$P = m\dot{R} = m\dot{x}, \quad E = \frac{\varepsilon}{\zeta\mu k}, \quad m = \frac{1}{\zeta k}, \quad \omega = \frac{k}{\sqrt{\mu}}.$$

It should be noted that $R \in [0, +\infty)$ hence $x \in [-R_0, +\infty)$. This circumstance is very important for further studies, first of all for analysis of quantum aspects of the theory.

Below we study the quantum mechanical properties of our lenses. One can perform the standard procedure of quantization. Then the conservation law (15) gives us the stationary Schrödinger equation for the spatial wave function $\Psi(x)$

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} + \left[-E + \frac{2\delta}{\zeta\mu k} + \frac{m\omega^2}{2} x^2 \right] \Psi = 0, \quad (16)$$

or, in the dimensionless form,

$$\frac{d^2\Psi}{dy^2} + (\varrho - y^2) \Psi = 0, \quad (17)$$

where

$$y = \sqrt{\frac{m\omega}{\hbar}} x, \quad \varrho = \frac{2}{\hbar\omega} \left(E - \frac{2\delta}{\zeta\mu k} \right).$$

However Eq. (17) is not purely the equation for the quantum harmonic oscillator, because the oscillator's wave functions are defined on the line $(-\infty, +\infty)$ whereas in the present case we have both the whole line and the half-line $y \in [-R_0\sqrt{m\omega/\hbar}, +\infty)$. The analytic continuation of y on the whole axis $(-\infty, +\infty)$ can be correctly explained only for the

AL type shells because they are acoustic wormholes as was mentioned above. Such a continuation of the spatial coordinate appears to be a somewhere artificial but necessary technique. Indeed, in the wormhole case we have matched the two non-embedded space-times, which both have their own infinitely distant points. Then after the continuation one can explicitly discriminate these spatial infinities from each other by virtue of a sign. Thus, besides the parameter δ , the ordinary and anomalous lenses have different topological properties. Below we distinguish these cases.

(i) *Anomalous lenses*. In this case physics admits the analytic continuation $y \in (-\infty, +\infty)$. For bound states the quantum boundary conditions, corresponding to the singular Stourm-Liouville problem, require $\Psi(+\infty) = \Psi(-\infty) = 0$, and the normalized solution of Eq. (17) can be expressed by means of the Hermite polynomials $H_n(y)$ [17]

$$\Psi(y) = \left(2^n \sqrt{\pi n!}\right)^{-1/2} \exp(-y^2/2) H_n(y), \quad (18)$$

where $n = 0, 1, 2, \dots$. The discrete values of energy are $\varrho = 2n + 1$ hence, taking into account Eq. (13),

$$E_n = \frac{2}{\zeta k(c_-^{-2} + c_+^{-2})} + \frac{\hbar k}{2\sqrt{c_-^{-2} + c_+^{-2}}}(2n + 1), \quad (19)$$

that indeed appears to be the energy of the quantum harmonic oscillator plus wormhole shift.

(ii) *Ordinary lenses*. In this case it is necessary to solve the Schrödinger equation (16) on the half-axis

$$R \in [0, +\infty) \Rightarrow y \in [-R_0 \sqrt{m\omega/\hbar}, +\infty).$$

Performing the transformation $z = y^2$ (which works like the baker's transformation [16]), we obtain that $z \in [0, +\infty)$. Then Eq. (17) can be written as the confluent hypergeometric equation

$$z \frac{d^2\varphi}{dz^2} + \left(\frac{3}{2} - z\right) \frac{d\varphi}{dz} + \frac{\varrho - 3}{4}\varphi = 0, \quad (20)$$

where $\Psi(z) = \exp(-z/2) \varphi(z)$. For bound states the quantum boundary conditions require $\Psi(0) = \Psi(+\infty) = 0$, thereby the confluent hypergeometric functions $\varphi(z)$ turn to be the Laguerre polynomials $L_n^{(\alpha)}(z)$, $\alpha > -1$ [15, 17]. Finally we obtain the normalized wave functions

$$\Psi(y) = \sqrt{\frac{n!}{\Gamma(n + 1/2)}} y \exp(-y^2/2) L_n^{(1/2)}(y^2), \quad (21)$$

where $n = 0, 1, 2, \dots$. Then we have the observable spectrum of energy to be determined by the expression $\varrho = 4n + 3$ hence, taking into account Eq. (13),

$$E_n = \frac{\hbar k}{2\sqrt{c_-^{-2} - c_+^{-2}}}(4n + 3), \quad (22)$$

which does not depend explicitly on the shell's fundamental constant ζ as it can easily be seen, but involves the constant k related to specific matter on the shell. Comparing expressions (19) and (22) we conclude that

$$E_{2n+1}^{(\text{AL})} - \frac{2}{\zeta k(c_-^{-2} + c_+^{-2})} = E_n^{(\text{OL})}, \quad (23)$$

that can be proved also by means of the relation between the Laguerre and Hermite polynomials.

In present paper the classical and the quantum aspects of the spherically symmetric thin shells in the motionless homogeneous superfluid helium were studied. We have considered

such singular hypersurfaces as the traversable interfaces between pairs of the domains, for instance, the phases “ $^3\text{He A} - ^3\text{He B}$ ”, the mixtures “ $^4\text{He} - ^3\text{He}$ ”, or the “inviscid impurity - He”. It was shown that these shells can give rise to the acoustic lenses which have to be sonic models of the composite space-time (i.e., the “patchwork manifold” or union space) consisting of regions with different fundamental constants.

Table 1: The classification of acoustic lenses into the ordinary (OL) and anomalous (AL) ones, the sign “ \star ” denotes the impossibility of the Lichnerowicz-Darmois-Israel’s junction.

$\sigma > 0$	$\epsilon_+ = \epsilon_-$		$\epsilon_+ = -\epsilon_-$	
	$\begin{pmatrix} \epsilon_+=1 \\ \epsilon_-=1 \end{pmatrix}$	$\begin{pmatrix} \epsilon_+=-1 \\ \epsilon_-= -1 \end{pmatrix}$	$\begin{pmatrix} \epsilon_+=1 \\ \epsilon_-= -1 \end{pmatrix}$	$\begin{pmatrix} \epsilon_+=-1 \\ \epsilon_-=1 \end{pmatrix}$
$c_+ > c_-$	OL	\star	\star	AL
$c_+ = c_-$	\star	\star	\star	AL
$c_+ < c_-$	\star	OL †	\star	AL

References

- [1] W. G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981).
- [2] T. A. Jacobson, Phys. Rev. D **44**, 1731 (1991); T. A. Jacobson and G. E. Volovik, Phys. Rev. D **58**, 064021 (1998).
- [3] W. G. Unruh, Phys. Rev. D **51**, 2827 (1995).
- [4] M. Visser, Class. Quantum Grav. **15**, 1767 (1998).
- [5] G. E. Volovik and T. Vachaspati, Int. J. Mod. Phys. B **10**, 471 (1996).
- [6] G. E. Volovik, Low Temp. Phys. **24**, 127 (1998); F. Wilczek, Int. J. Mod. Phys. A **13**, 863 (1998).
- [7] G. Dautcourt, Math. Nachr. **27**, 277 (1964).
- [8] W. Israel, Nuovo Cimento B **44**, 1 (1966).
- [9] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973); C. Barrabés and G. F. Bressange, Class. Quantum Grav. **14**, 805 (1997).
- [10] K. G. Zloshchastiev, Mod. Phys. Lett. A **13**, 1419 (1998).
- [11] G. E. Volovik, JETP Lett. **67**, 698 (1998); Czech. J. Phys. **46**, 3048 (1996).
- [12] P. Laguna-Castillo and R. A. Matzner, Phys. Rev. D **34**, 2913 (1986); V. A. Berezin, V. A. Kuzmin, and I. I. Tkachev, Phys. Rev. D **36**, 2919 (1987).
- [13] V. A. Berezin, V. A. Kuzmin, and I. I. Tkachev, Phys. Lett. B **120**, 91 (1983); V. A. Berezin, N. G. Kozimirov, V. A. Kuzmin, and I. I. Tkachev, Phys. Lett. B **212**, 415 (1988); D. S. Goldwirth and J. Katz, Class. Quantum Grav. **12**, 769 (1995).
- [14] M. Visser, *Lorentzian wormholes – from Einstein to Hawking* (AIP, New York, 1995).
- [15] K. G. Zloshchastiev, Phys. Rev. D **57**, 4812 (1998).
- [16] F. C. Moon, *Chaotic Vibrations* (Wiley, New York, 1987).
- [17] E. Janke, F. Emde, and F. Lösch, *Tafeln Höherer Funktionen* (Teubner, Stuttgart, 1960).